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The Center Conditions for a Liénard System

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Abstract—We obtain necessary and sufficient coefficient conditions for a center at the origin for a Liénard system with nonlinearities of degree six. The necessity of the conditions is derived from the first seven focus quantities and their sufficiency is proved by Cherkas's method.

Keywords—Center problem, Liénard system, Polynomial systems, Local integrability.

1. INTRODUCTION

Consider the system

$$\frac{dx}{dt} = y + \sum_{s=2}^{n} X_s(x,y) = X(x,y), \qquad \frac{dy}{dt} = -x + \sum_{s=2}^{n} Y_s(x,y) = Y(x,y), \tag{1}$$

where $X_s(x,y)$, $Y_s(x,y)$ are polynomials in x and y of degree s. Conversion to polar coordinates shows that near the origin either all nonstationary trajectories of (1) are ovals (in which case the origin is called a *center*) or they are all spirals (in which case the origin is called a *focus*). The problem of distinguishing between centers and foci is known as the *Poincaré center problem.* Generally speaking, resolving the center problem is the first step toward the investigation of the so-called *cyclicity problem*, which is also known as the local 16^{th} Hilbert problem (see, e.g., [1,2]). A criterion for distinguishing between a center and a focus of (1) is given by the next theorem, which is due to Poincaré and Lyapunov.

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THEOREM 1. *The origin of system (1) is a singular point of the center type if and only if the system admits a first integral of the form*

$$x^2 + y^2 + \sum_{k=3}^{\infty} p_k(x,y) \equiv c, \tag{2}$$

where $p_k(x,y)$ are homogeneous polynomials of degree k.

It is readily seen that it is always possible to find a function of the form

$$\Psi(x,y) = x^2 + y^2 + \sum_{s=3}^{\infty} \sum_{j=0}^{s} v_{j,s-j} x^j y^{s-j}, \tag{3}$$

such that

$$\frac{\partial \Psi}{\partial x} X(x,y) + \frac{\partial \Psi}{\partial y} Y(x,y) = \sum_{i=1}^{\infty} g_i \left(x^2 + y^2\right)^{i+1}, \tag{4}$$

where $v_{j,s-j}$ and g_i are polynomials in coefficients of X and Y. We call the polynomials g_i the *focus quantities* of system (1). If for some values of parameters of (1) $g_1 = g_2 = \cdots = 0$ then the corresponding system (1) has a formal integral (3); however, this yields the existence of a convergent integral of form (3) (see, e.g., [3, Chapter 1, Section 6]) and, therefore, the existence of a center at the origin of system (1). Otherwise, if there exists k such that $g_k \neq 0$ then the corresponding system of differential equations has a focus at the origin.

For a given polynomial ideal $I = \langle f_1, \dots, f_p \rangle \subset k[x_1, \dots, x_n]$ we denote by $\mathbf{V}(I)$ the variety of I, that is the zero set of all polynomials in I. A possible way to distinguish between a center and a focus of system (1) is as follows. One computes few first focus quantities, say $g_1, \dots, g_m$, and then tries to check whether the variety defined by these quantities coincides with the variety of the ideal of all focus quantities, that is, whether

$$\mathbf{V}(\langle g_1, \dots, g_m \rangle) = \mathbf{V}(\langle g_1, g_2, \dots \rangle). \tag{5}$$

Due to the Hilbert basis theorem equality (5) must hold for some m; however, there are no regular methods for verifying (5). According to Theorem 1 in order to prove (5) it is sufficient to show that all systems from $\mathbf{V}(\langle g_1, \dots, g_m \rangle)$ admit a first integral of form (2). However, no regular methods for constructing such an integral are known as well, and this is one of the reasons why: despite its hundred-year history, the center problem has been solved for only a few subfamilies of system (1), mainly subfamilies of the so-called cubic system, that is, system of the form

$$\frac{dx}{dt} = y + X_2(x,y) + X_3(x,y), \qquad \frac{dy}{dt} = -x + Y_2(x,y) + Y_3(x,y) \tag{6}$$

(see, for example, [4–14] and references therein).

Consider now a subfamily of (1) of the form

$$\dot{x} = y, \qquad \dot{y} = -g(x) - y f(x), \tag{7}$$

where $f(x)$ and $g(x)$ are real polynomials, such that

$$g(0) = 0, \qquad g'(0) > 0. \tag{8}$$

If the origin is a nondegenerate center or a focus of system (7) then (8) holds. System (7) is called the *Liénard system.*

Unlike for the general polynomial system (1), for system (7) there is a regular method to verify whether it has a center at the origin. Namely, let $F(x)$, $G(x)$ be the antiderivatives of $f(x)$ and $g(x)$, respectively,

$$F(x) = \int_0^x f(s)\,ds, \qquad G(x) = \int_0^x g(s)\,ds. \tag{9}$$

Then the following theorem of Cherkas [6] (see also [8,9]) provides a simple efficient way for proving the existence of a center in system (7).

THEOREM 2. *System (7) has a center at the origin if and only if there exists a function $z(x)$ which is defined and analytic on a neighborhood of the origin, satisfies the condition*

$$z(0) = 0, \qquad z'(0) < 0, \tag{10}$$

and solves the system of equations

$$F(x) = F(z), \qquad G(x) = G(z). \tag{11}$$

In the present paper we apply this theorem to give coefficient conditions for a center at the origin for a polynomial system of degree six, which contains a rather large subfamily of the cubic system (6).

2. THE CENTER CONDITIONS

First we note that the system

$$\dot{x} = p_3(x)y, \qquad \dot{y} = -p_0(x) - p_1(x)y - p_2(x)y^2, \tag{12}$$

where $p_0(0) = 0,\ p_3(0)p_0'(0) > 0,\ p_3(x) \not\equiv 0$, can be transformed to a Liénard system (7). Namely, using the substitution

$$x_1 = x, \qquad y_1 = y\psi(x), \tag{13}$$

with $\psi(x) = \exp(\int_0^x p_2(t)/p_3(t)\,dt)$, and then rescaling by $p_3(x)/\psi(x)$ and dropping the subscript 1 in x_1 and y_1 we obtain from (12) the system

$$\dot{x} = y, \qquad \dot{y} = -\frac{p_0(x)\psi^2(x)}{p_3(x)} - \frac{p_1(x)\psi(x)}{p_3(x)}y.$$

Let

$$G(x) = \int_0^x \frac{p_0(t)}{p_3(t)}\psi^2(t)\,dt, \qquad F(x) = \int_0^x \frac{p_1(t)}{p_3(t)}\psi(t)\,dt.$$

As a consequence of Theorem 2 we have the following proposition and its corollary.

THEOREM 3. *(See [6].) The origin is a center for system (12) if and only if either each of the equations*

$$\begin{aligned} \frac{R(x)}{p_1^3(x)} &= \frac{R(y)}{p_1^3(y)}, \\ \frac{Q(x)}{R^2(x)} &= \frac{Q(y)}{R^2(y)}, \end{aligned} \tag{14}$$

where $R(x) = p_3(p_0'p_1 - p_0p_1') + p_0p_1p_2$, $Q(x) = p_3(p_0'p_1R - R'p_0p_1 + 2Rp_0p_1') + p_0p_1p_2R$, defines the same holomorphic function $y = y(x)$ such that $y(0) = 0$, $y'(0) = -1$, or one of equations (14) is the identity.

COROLLARY 1. *If*

$$R(x) \equiv cp_1^3(x), \tag{15}$$

where c is a constant, then the origin of (12) is a center.

If the functions p_i $(i = 0, 1, 2, 3)$ in (12) satisfy condition (15) we say that the system has a *trivial* center at the origin.

As we have mentioned above, in recent years many studies have been devoted to the center problem for system (6). It is not difficult to see that a large subfamily of (6) can be transformed to a system of the Liénard type. Indeed, an orthogonal transformation of the phase plane brings (6) to the system of the same form but with the coefficient of y^2 in X_2 equal to zero [13]. Setting in X_3 the coefficients of the terms xy^2 and y^3 equal to zero we obtain a family of cubic systems of codimension two in the family (6) of cubic systems. Namely, we obtain the system

$$\begin{aligned} \dot{x} &= y\left(1 + b_1 x + b_2 x^2\right) + Sx^2 + Rx^3, \\ \dot{y} &= -x\left(1 + u_1 x + u_2 x^2\right) + \left(Vx + Wx^2\right) y + (Gx + T)y^2 + Ey^3, \end{aligned} \tag{16}$$

which can be transformed to a Liénard system of form (7) [4]. Thus, the solution of the center problem for this Liénard system will also provide the solution of the center problem for a large subfamily of cubic system (6). However, finding the center conditions for (16) (or the corresponding Liénard system) involves very laborious calculations which we were unable to complete using our present computer facilities, and therefore we will limit our study to the subfamily of (16) where $b_2 = 0$, $b_1 \neq 0$, $G = E = 0$. Note that this subfamily contains the cubic system with the center of cyclicity 11 studied by Żołądek [14].

The transformation $x \to x/b_1$, $y \to y/b_1$ allows us to set in (16) $b_1 = 1$. Therefore we will consider the system

$$\dot{x} = yP_2(x) + P_3(x), \qquad \dot{y} = Q_3(x) + Q_2(x)y + Q_1 y^2, \tag{17}$$

where $P_2 = x + 1$, $P_3 = Sx^2 + Rx^3$, $Q_3 = -x(1 + u_1 x + u_2 x^2)$, $Q_2 = Vx + Wx^2$, $Q_1 = T$.

Making use of the substitution $y \to (y - P_3)/P_2$ we obtain from (17), after a rescaling of time, the system

$$\dot{x} = (x + 1)y, \qquad \dot{y} = R_6 + R_3 y + (T + 1)y^2, \tag{18}$$

where

$$R_6 = -x\left(1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5\right), \qquad R_3 = A_0 x + \tilde{A}x^2 + \tilde{L}x^3, \tag{19}$$

and

$$\begin{aligned} a_1 &= 2 + u_1, & a_2 &= 1 + 2u_1 + u_2 + SV, \\ a_3 &= -S^2T + u_1 + 2u_2 + RV + S(V + W), & a_4 &= u_2 + SW + R(V - 2ST + W), \\ a_5 &= RW - R^2T, & A_0 &= 2S + V, \\ \tilde{A} &= (3R + S - 2ST + V + W), & \tilde{L} &= 2R - 2RT + W. \end{aligned} \tag{20}$$

We will consider the general case when the coefficients $a_1, \ldots, a_5, A_0, \tilde{A}, \tilde{L}$ of system (18) are not defined by formulae (20), but are arbitrary parameters. In order to obtain the center conditions for this system we will need the following property.

PROPOSITION 1. *If T is an irrational number then system (18) can have only a trivial center at the origin.*

PROOF. Substitution (13), which in this case is $y \mapsto y(x+1)^{T+1}$, and the rescaling of time by $(x+1)^{T+2}$ bring (18) to the form

$$\dot{x} = y, \qquad \dot{y} = R_6(x+1)^{-2T-3} + R_3(x+1)^{-T-2}y.$$

Computing functions (9) we obtain

$$\begin{aligned} F(x) &= \int_0^x f(s)\,ds = -\tilde{R}_3(x)(x+1)^{-T-1} + \tilde{R}_3(0), \\ G(x) &= \int_0^x g(s)\,ds = -\tilde{R}_6(x)(x+1)^{-2T-2} + \tilde{R}_6(0). \end{aligned} \tag{21}$$

Therefore system (11) in this case has the form

$$\tilde{R}_3(x)(x+1)^{-T-1} = \tilde{R}_3(z)(z+1)^{-T-1}, \qquad \tilde{R}_6(x)(x+1)^{-2T-2} = \tilde{R}_6(z)(z+1)^{-2T-2}, \tag{22}$$

where $\tilde{R}_3$ and $\tilde{R}_6$ are some polynomials.

It follows from (21) that the first of equations (22) can be the identity only if $R_3 \equiv 0$. It is easily seen that in such a case the system has a trivial center at the origin. From (19) and (21) it is clear that the second of equations (22) cannot be the identity. So, we now assume that neither of the equations in (21) is the identity. Then, dividing the second equation by the squared first equation, we obtain from (21) the equivalent system

$$\begin{aligned} \frac{\tilde{R}_6(x)}{\tilde{R}_3^2(x)} &= \frac{\tilde{R}_6(z)}{\tilde{R}_3^2(z)}, \\ \tilde{R}_3(x)(x+1)^{-T-1} &= \tilde{R}_3(z)(z+1)^{-T-1}. \end{aligned} \tag{23}$$

After the substitution $z = p(x+1) - 1$ the second of equations (23) takes the form

$$R(p, x) = p^{-T-1}, \tag{24}$$

where R is a rational function. Thus, system (23) has no solution of the form $z = -x + \cdots$, because the first of equations (23) defines an algebraic curve, whereas equation (24) defines a transcendental curve (note that the condition $z = -x + \cdots$ yields that $p(x)$ in (24) is not a constant). ∎

THEOREM 4. *System (18) with $A_0 = 0$ has a center at the origin if and only if one of the following conditions holds:*

(1) $\tilde{A} = \tilde{L} = 0$;
(2) $\tilde{A} = T - 3 = a_1 - 6 = 2a_3 - 2a_4 + a_5 - 16 = 8a_2 - 2a_4 + a_5 - 96 = 0$;
(3) $\tilde{A} = T + 2 = a_1 + 1 = a_2 + a_3 = a_4 + a_5 = 0$;
(4) $\tilde{A} = 2T - 1 = 2a_1 - 5 = a_2 - 2a_4 + 4a_5 - 2 = 2a_3 - 6a_4 + 12a_5 - 1 = 0$.

PROOF. To prove the theorem we computed by formula (4) the first seven focus quantities of system (18). The first quantity is

$$g_1 = \frac{\left(\tilde{A} + A_0 - a_1 A_0 + T A_0\right)}{4}. \tag{25}$$

The other focus quantities, $g_2, \ldots, g_7$, contain, respectively, 28, 118, 388, 1079, 2662, 5989 terms, so it does not make sense to present these quantities here. Instead, in the Appendix we give a

MATHEMATICA code, which we used to compute these quantities. Setting $A_0 = 0$ in the polynomials $g_1, g_2, \ldots, g_7$ and computing the minimal associate primes of the ideal $I = \langle g_1, g_2, \ldots, g_7 \rangle$ by means of the routine *minAssChar* of Singular [15] (*minAssChar* finds associate primes of a polynomial ideal using the characteristics sets [16]) we obtain the four series of conditions presented in the statement of the proposition. Thus (1)–(4) are the necessary center conditions for system (18).

To prove that they are the sufficient center conditions we use Theorem 3. Substituting the corresponding values of the parameters into (14) we find that if Condition (1) is fulfilled then both equations (14) are identities, if Condition (2) is satisfied then each of the equations (14) has the solution $y = -x/(1+2x)$, if Condition (3) takes place then (14) has the solution $y = -x$, and, finally, when Condition (4) holds then both equations (14) are satisfied by $y = -x/(1+x)$. ∎

Consider now the case $A_0 \neq 0$. We observe that equations (14) depend on $A = \tilde{A}/A_0$, $L = \tilde{L}/A_0$. Therefore without loss of generality it is sufficient to study only the case $A_0 = 1$.

THEOREM 5. *System (18) with $A_0 = 1$ has a center at the origin if and only if either*

(α) $a_1 = 1+\tilde{A}+T$, $a_2 = (2\tilde{A}-4\tilde{A}^2+2\tilde{L}+2\tilde{A}\tilde{L}+6\tilde{L}^2-3\tilde{A}T-2\tilde{A}^2T+8\tilde{A}\tilde{L}T-3\tilde{A}T^2+2\tilde{A}^2T^2+4T^2+3\tilde{A}T^3)/d_0$, $a_3 = (1+T)(2\tilde{L}-4\tilde{A}\tilde{L}+6\tilde{L}^2-2\tilde{A}^2T-4\tilde{L}T+5\tilde{A}\tilde{L}T+\tilde{A}^2T^2+2\tilde{L}T^2)/d_0$, $a_4 = \tilde{L}T(1+T)(-3\tilde{A}+3\tilde{L}+2\tilde{A}T)/d_0$, $a_5 = \tilde{L}^2T(T^2-1)/d_0$, *where* $d_0 = 2-4\tilde{A}+6\tilde{L}-3T+2\tilde{A}T+T^2 \neq 0$, *or*

(β) *the vector of the coefficients* $(T, \tilde{A}, \tilde{L}, a_1, a_2, a_3, a_4, a_5)$ *has one of the following values:*

(1) $(-4, -1, -2, -4, s, -(10/3)(s-4), (4/3)(s-4), (8/3)(s-4))$;

(2) $(-4, 0, 0, -3, s, -(5/3)s, -(7/2)s, -(3/2)s)$;

(3) $(-3, -1, 0, -3, s, 8-3s, 3(s-3), 3-s)$;

(4) $(-2/3, 2/3, -1/3, 1, s, (5/9)s, -(7/18)s, (1/18)s)$;

(5) $(-1/3, 1/3, 0, 1, s, (1/9)s, 0, 0)$;

(6) $(1/3, 1, 0, 7/3, s, (1/27)(39s-56), (13/81)(3s-5), (1/81)(3s-5))$;

(7) $(1/3, 2/3, 0, 2, s, (8/9)(s-1), 0, 0)$;

(8) $(2/3, 1, 0, 8/3, s, (2/27)(21s-40), (16/81)(3s-7), 0)$;

(9) $(2/3, 4/3, 1/3, 3, s, (1/9)(19s-48), (11/9)(s-3), (1/9)(s-3))$;

(10) $(-4/3, 1/3, -2/3, 0, s, -(2/9)s, -(4/9)s, (8/45)s)$;

(11) $(-4/3, 0, 0, s, -1/3, -(7/9)s, (11/54)s, -s/54)$;

(12) $(5/2, 2, 1, 11/2, s, (1/6)(23s-198), (1/6)(29s-306), (1/2)(4s-45))$;

(13) $(7/3, 2, 1, 16/3, s, (2/27)(51s-416), (1/162)(771s-7712), (2/3)(3s-32))$;

(14) $(7/3, 5/3, 0, 5, s, (1/9)(29s-200), 3s-25, (1/5)(3s-25))$;

(15) $(4, 0, 0, 5, s, s, s/2, s/10)$;

(16) $(4, 2, 1, 7, s, (1/3)(13s-168), (13/3)(s-15), s-15)$;

(17) $(4, 3, 2, 8, s, 6s-112, 4(3s-68), 8(s-24))$;

(18) $(5, 2, 1, 8, s, (14/3)(s-16), (17/6)(s-16), (2/3)(s-16))$;

(19) $(5, 3, 0, 9, s, 19s/3-144, 11s-297, 3s-81)$;

(20) $(5/4, 3/2, 9/16, 15/4, s, (1/48)(124s-495), (1/128)(284s-1395), (81/512)(4s-21))$;

(21) $(-1/4, 1/2, 1/16, 5/4, s, (5/48)(4s-1), (5/384)(4s-1), (1/1536)(4s-1))$;

(22) $(7/5, 8/5, 16/25, 4, s, (2/5)(7s-32), (1/50)(131s-736), (512/625)(s-6))$;

(23) $(4/5, 6/5, 9/25, 3, s, (1/15)(29s-72), (27/25)(s-3), (81/625)(s-3))$;

(24) $(2, 1, 0, 4, s, 2(s-4), w, 0)$;

(25) $(1/5, 4/5, 4/25, 2, s, (16/15)(s-1), (16/75)(s-1), (32/1875)(s-1))$;

(26) $(-3, 0, 0, -2, s, -(4/3)s, -(4/3)s, 0)$;

(27) $(-3, 2, 1, 0, s, 2s, (3/2)s, (2/5)s)$;

(28) $(-3/2, 0, 0, -1/2, s, -(5/6)s, (1/6)s, 0)$;

(29) $(-1, v, v-1, v, (1/3)(5v-2)(1-v+s), (1/3)(v+2v^2-3)(1-v+s), (2/3)(v-1)^2(1-v+s))$;

(30) $(-1, 1, 0, 1, s, s, w, w)$;

(31) $(5/3, 4/3, 0, 4, s, (2/9)(11s-48), (3/2)s-8, 0)$;

(32) $(2, w, 0, 3+w, s, (1/3)(s-15w^2+w(5s-9)), (1/3)w(s-6w^2+w(2s-9)), 0)$;

(33) $(3, w, 0, 4+w, s, (1/3)(2s-20w-20w^2+5ws), -(1/6)(1+4w)(10w+4w^2-s-ws), (1/6)w(s-10w-4w^2+ws))$;

(34) $(3, 2, v, 6, s, 4(s-10), w, 2(48-4s+w))$;

(35) $(-2, 0, v, -1, s, -s, -w, w)$;

(36) $(-2, 1+v, v, v, s, (1/3)s(2+5v), (1/6)sv(7+4v), sv^2/2)$;

(37) $(1/2, 1, v, 5/2, s, (1/2)(3s-5), w, (1/4)(2w-s+2))$;

(38) $(-2/5, 2/5, 1/25, 1, s, s/50, s/1250)$;

(39) $(4/3, 5/3, 2/3, 4, s, (2/9)(13s-60), (1/9)(25s-141), (8/9)(s-6))$;

(40) $(0, w, (1/3)(2w-1), 1+w, s, (1/9)(-4-25w^2-3s+w(16+15s)), (1/9)(1-w-2w^2)(5w-3s-1), (1/27)(1-2w)^2(1-5w+3s))$;

(41) $(1, w, w/3, 2+w, s, (5/9)w(3s-7w), (w/9)(3s-14w^2+w(6s-13)), -(2/27)w^2(3+7w-3s))$,

where s, v, and w are arbitrary real parameters.

To prove the theorem we need the following result.

LEMMA 1. *(See [7].) If system (18) with $A_0 = 1$ has a trivial center at the origin then either T is rational or Condition (α) of Theorem 5 holds.*

The lemma follows from Theorem 3 of [7]. The latter theorem gives the necessary and sufficient conditions of existence a trivial center in system (18). Namely, it is shown that the system has a center if and only if one of eight sets of conditions for the coefficients is fulfilled. One of these sets of conditions contains $A_0 = 0$ (this contradicts our assumption $A_0 = 1$), six sets of the conditions require that T is an integer (T can be equal to -2, -1, 0, 1, 2 or 3), and the remaining set coincides with (α) of Theorem 5 when $A_0 = 1$ (see [7] for the details).

PROOF OF THEOREM 5. According to Corollary 1 system (18) has a center at the origin if there exists a constant c such that

$$(x+1)\left(R_6' R_3 - R_6 R_3'\right) - R_6 R_3 (T+1) \equiv c R_3^3. \tag{26}$$

Equating the coefficients of the same powers of x on both sides of (26) we obtain the system of polynomial equations

$$f_1 = f_2 = \cdots = f_8 = 0, \tag{27}$$

where $f_1 = -\tilde{L}(2a_5 - a_5T + c\tilde{L}^2)$, $f_2 = -3a_5\tilde{A} + a_5T\tilde{A} - a_4\tilde{L} - 3a_5\tilde{L} + a_4T\tilde{L} - 3c\tilde{A}\tilde{L}^2$, $f_3 = \tilde{A} + (1 - a_1 + T)$, $f_4 = 2\tilde{A} + T\tilde{A} + 2\tilde{L} - 2a_2 + a_1T - c$, $f_5 = -a_3\tilde{A} - 3a_4\tilde{A} + a_3T\tilde{A} - c\tilde{A}^3 + a_2\tilde{L} - a_3\tilde{L} + a_2T\tilde{L} - 3a_4 - 5a_5 1 + a_4T - 6c\tilde{A}\tilde{L}$, $f_6 = -2a_3\tilde{A} + a_2T\tilde{A} + 2a_1\tilde{L} + a_1T\tilde{L} - 2a_3 - 4a_4 + a_3T - 3c\tilde{A}^2 - 3c\tilde{L}$, $f_7 = a_1\tilde{A} - a_2\tilde{A} + a_1T\tilde{A} + 3\tilde{L} + a_1\tilde{L} + T\tilde{L} - a_2 - 3a_3 + a_2T - 3c\tilde{A}$, $f_8 = -2a_4\tilde{A} - 4a_5\tilde{A} + a_4T\tilde{A} - 2a_4\tilde{L} + a_3T\tilde{L} - 3c\tilde{A}^2\tilde{L} - 4a_5 + a_5T - 3c\tilde{L}^2$.

Resolving (27) with respect to the variables a_i, $i = \overline{1,5}$ and c we obtain Condition (α) of the theorem.

To obtain Conditions (β) we use the first seven focus quantities $g_1, \dots, g_7$ with $A_0 = 1$. Then from (25) we obtain

$$a_1 = 1 + T + \tilde{A}. \tag{28}$$

We substitute this expression into the other focus quantities and then from $g_2 = 0$ we find that

$$a_3 = \frac{\left(-a_2 + a_2T + \tilde{A} + 5a_2\tilde{A} - T\tilde{A} - 2T^2\tilde{A} - 5\tilde{A}^2 - 5T\tilde{A}^2 + 4\tilde{L} + 2T\tilde{L} - 5\tilde{A}\tilde{L}\right)}{3} \tag{29}$$

and from the equation $g_3 = 0$ we have

$$\begin{aligned} a_5 = \Big(&8a_2 - 75a_4 - 8a_2T - 15a_4T - 2a_2T^2 + 2a_2T^3 - 8\tilde{A} - 51a_2\tilde{A} + 105a_4\tilde{A} + 8T\tilde{A} \\ &+35a_2T\tilde{A} + 18T^2\tilde{A} + 4a_2T^2\tilde{A} - 2T^3\tilde{A} - 4T^4\tilde{A} + 51\tilde{A}^2 + 105a_2\tilde{A}^2 - 19T\tilde{A}^2 \\ &-35a_2T\tilde{A}^2 - 74T^2\tilde{A}^2 - 4T^3\tilde{A}^2 - 105\tilde{A}^3 - 70a_2\tilde{A}^3 - 35T\tilde{A}^3 + 70T^2\tilde{A}^3 + 70\tilde{A}^4 \\ &+70T\tilde{A}^4 - 8\tilde{L} + 84a_2\tilde{L} - 4T\tilde{L} + 42a_2T\tilde{L} - 16T^2\tilde{L} - 8T^3\tilde{L} - 33\tilde{A}\tilde{L} - 105a_2\tilde{A}\tilde{L} \\ &-161T\tilde{A}\tilde{L} - 46T^2\tilde{A}\tilde{L} + 140T\tilde{A}^2\tilde{L} + 70\tilde{A}^3\tilde{L} - 84\tilde{L}^2 - 42T\tilde{L}^2 + 105\tilde{A}\tilde{L}^2\Big)\frac{1}{45}. \end{aligned} \tag{30}$$

Substituting these expressions of a_1, a_3, a_5 into the remaining quantities we obtain the system of polynomial equations

$$g_4 = g_5 = g_6 = g_7 = 0 \tag{31}$$

in the variables T, $\tilde{A}$, $\tilde{L}$, a_2, a_4. The equation $g_4 = 0$ yields

$$\begin{aligned} a_4 = \Big(&472a_2 - 192a_2T - 382a_2T^2 + 32a_2T^3 + 66a_2T^4 + 4a_2T^5 - 472\tilde{A} - 3856a_2\tilde{A} + 192T\tilde{A} \\ &+1008a_2T\tilde{A} + 1326T^2\tilde{A} + 1716a_2T^2\tilde{A} + 528T^3\tilde{A} + 28a_2T^3\tilde{A} - 270T^4\tilde{A} - 48a_2T^4\tilde{A} \\ &-144T^5\tilde{A} - 8T^6\tilde{A} + 3856\tilde{A}^2 + 11977a_2\tilde{A}^2 - 64T\tilde{A}^2 - 1575a_2T\tilde{A}^2 - 6980T^2\tilde{A}^2 \\ &-2268a_2T^2\tilde{A}^2 - 2920T^3\tilde{A}^2 - 70a_2T^3\tilde{A}^2 + 244T^4\tilde{A}^2 + 104T^5\tilde{A}^2 - 11977\tilde{A}^3 - 17591a_2\tilde{A}^3 \\ &-4249T\tilde{A}^3 + 735a_2T\tilde{A}^3 + 11886T^2\tilde{A}^3 + 854a_2T^2\tilde{A}^3 + 4186T^3\tilde{A}^3 + 28T^4\tilde{A}^3 \\ &+17591\tilde{A}^4 + 12110a_2\tilde{A}^4 + 11571T\tilde{A}^4 - 7644T^2\tilde{A}^4 - 1624T^3\tilde{A}^4 - 12110\tilde{A}^5 - 3080a_2\tilde{A}^5 \\ &-10570T\tilde{A}^5 + 1540T^2\tilde{A}^5 + 3080\tilde{A}^6 + 3080T\tilde{A}^6 - 472\tilde{L} + 5112a_2\tilde{L} - 516T\tilde{L} \\ &+4572a_2T\tilde{L} - 1100T^2\tilde{L} + 1224a_2T^2\tilde{L} - 1040T^3\tilde{L} + 108a_2T^3\tilde{L} - 312T^4\tilde{L} - 16T^5\tilde{L} \\ &-1256\tilde{A}\tilde{L} - 18333a_2\tilde{A}\tilde{L} - 8172T\tilde{A}\tilde{L} - 9765a_2T\tilde{A}\tilde{L} - 3228T^2\tilde{A}\tilde{L} - 1008a_2T^2\tilde{A}\tilde{L} \\ &+656T^3\tilde{A}\tilde{L} + 192T^4\tilde{A}\tilde{L} + 6356\tilde{A}^2\tilde{L} + 21924a_2\tilde{A}^2\tilde{L} + 25956T\tilde{A}^2\tilde{L} + 5292a_2T\tilde{A}^2\tilde{L} \\ &+7938T^2\tilde{A}^2\tilde{L} - 308T^3\tilde{A}^2\tilde{L} - 4333\tilde{A}^3\tilde{L} - 8820a_2\tilde{A}^3\tilde{L} - 25851T\tilde{A}^3\tilde{L} - 4046T^2\tilde{A}^3\tilde{L} \\ &-3290\tilde{A}^4\tilde{L} + 8820T\tilde{A}^4\tilde{L} + 3080\tilde{A}^5\tilde{L} - 5112\tilde{L}^2 + 2268a_2\tilde{L}^2 - 4572T\tilde{L}^2 + 1134a_2T\tilde{L}^2 \\ &-1224T^2\tilde{L}^2 - 108T^3\tilde{L}^2 + 16065\tilde{A}\tilde{L}^2 - 2835a_2\tilde{A}\tilde{L}^2 + 6363T\tilde{A}\tilde{L}^2 - 126T^2\tilde{A}\tilde{L}^2 - 19089\tilde{A}^2\tilde{L}^2 \\ &-2457T\tilde{A}^2\tilde{L}^2 + 8820\tilde{A}^3\tilde{L}^2 - 2268\tilde{L}^3 - 1134T\tilde{L}^3 + 2835\tilde{A}\tilde{L}^3\Big)\frac{1}{(21d)}, \end{aligned} \tag{32}$$

where

$$d = 176 + 144T + 36T^2 + 4T^3 - 597\tilde{A} - 315T\tilde{A} \\ -42T^2\tilde{A} + 645\tilde{A}^2 + 165T\tilde{A}^2 - 220\tilde{A}^3 + 108\tilde{L} + 54T\tilde{L} - 135\tilde{A}\tilde{L}. \tag{33}$$

Expression (32) is defined if the denominator d is different from zero. If it is equal to zero then there are two possibilities,

$$5\tilde{A} - 2T - 4 = 0, \tag{34}$$

$$5\tilde{A} - 2T - 4 \neq 0 \tag{35}$$

(in the first case d does not depend on $\tilde{L}$, and in the second case it is linear in $\tilde{L}$). Assume (34) takes place. Then it yields $T = (5\tilde{A} - 4)/2$. Substituting this value into the polynomials g_4, g_5, g_6, g_7 and computing with *minAssChar* we find that the minimal associate primes of the ideal $\langle g_4, g_5, g_6, g_7 \rangle$ are: $J_1 = \langle \tilde{A} \rangle$, $J_2 = \langle \tilde{A} - 1 \rangle$, $J_3 = \langle \tilde{A} - 2 \rangle$, $J_4 = \langle h_1, \ldots, h_7 \rangle$, where $h_1 = 40\tilde{A}^3 - 30\tilde{A}^2\tilde{L} - 36a_2\tilde{A} - 156\tilde{A}^2 - 3a_2\tilde{L} + 78\tilde{A}\tilde{L} + 3\tilde{L}^2 + 24a_2 + 15a_4 + 40\tilde{A} - 36\tilde{L}$ and $h_2, \ldots, h_7$ are some polynomials in a_2, a_4, $\tilde{A}$, $\tilde{L}$, T. The ideals J_1, J_2, J_3 yield Conditions (35), (37), (34) of the theorem, respectively. The equation $h_1 = 0$ gives $a_4 = (-24a_2 - 40\tilde{A} + 36a_2\tilde{A} + 156\tilde{A}^2 - 140\tilde{A}^3 + 36\tilde{L} + 3a_2\tilde{L} - 78\tilde{A}\tilde{L} + 30\tilde{A}^2\tilde{L} - 3\tilde{L}^2)/15$. If $\tilde{L} \neq 1/8(-16 + 34\tilde{A} - 15\tilde{A}^2)$ using this value of a_4 and *minAssChar* we obtain particular cases of (α). If $\tilde{L} = 1/8(-16+34\tilde{A}-15\tilde{A}^2)$ then we have particular cases of Conditions (41), (34), (40), (32) of the theorem.

In the case when (35) holds we obtain

$$\tilde{L} = -\frac{176 + 144T + 36T^2 + 4T^3 - 597\tilde{A} - 315T\tilde{A} - 42T^2\tilde{A} + 645\tilde{A}^2 + 165T\tilde{A}^2 - 220\tilde{A}^3}{27\left(4 + 2T - 5\tilde{A}\right)}. \tag{36}$$

Resolving the system $g_4 = g_5 = 0$ with respect to a_2 and a_4 we obtain a particular case of Condition (α).

Consider now the case when expression (32) is defined, that is, its denominator is different from zero. We first reduce polynomials g_k modulo the ideals of the previous polynomials, that is,

$$\hat{g}_6 := g_6 \bmod \langle g_5 \rangle, \qquad \hat{g}_7 := g_7 \bmod \langle g_5, g_6 \rangle$$

(one can do this with, for example, *PolynomialReduce* of MATHEMATICA or *reduce* of Singular). Substituting the above value of a_4 into $\hat{g}_5$, $\hat{g}_6$, $\hat{g}_7$ we see that the obtained expressions are rational functions, such that the numerators are factored in two terms (note that if we use g_6, g_7 instead of the reduced polynomials then the factorization is not possible). One of these terms is the same for all numerators and is equal to $f := 2a_2 - 3a_2T + a_2T^2 - 2\tilde{A} - 4a_2\tilde{A} + 3T\tilde{A} + 2a_2T\tilde{A} + 3T^2\tilde{A} - 2T^3\tilde{A} + 4\tilde{A}^2 + 2T\tilde{A}^2 - 2T^2\tilde{A}^2 - 2\tilde{L} + 6a_2\tilde{L} - 4T^2\tilde{L} - 2\tilde{A}\tilde{L} - 8T\tilde{A}\tilde{L} - 6\tilde{L}^2$. The second terms are different but depend only on $\tilde{A}$, $\tilde{L}$, and $\tilde{T}$.

Note that along with (28)–(32) the equation $f = 0$ yields Condition (α) of the theorem, so we can assume that $f \neq 0$. Denote by $\bar{g}_5$, $\bar{g}_6$, $\bar{g}_7$ the denominators of $\hat{g}_5/f$, $\hat{g}_6/f$, $\hat{g}_7/f$, respectively. We compute the resultants Γ_1 of the polynomials $\bar{g}_5$, $\bar{g}_6$, Γ_2 of $\bar{g}_6$, $\bar{g}_7$, and Γ_3 of $\bar{g}_5$, $\bar{g}_7$ with respect to $\tilde{L}$. Γ_1, Γ_2, and Γ_3 have the greatest common divisor $g = (T-3)(2+T)(4+2T-5\tilde{A})^4(236 + 258T + 78T^2 + 4T^3 - 840\tilde{A} - 588T\tilde{A} - 84T^2\tilde{A} + 987\tilde{A}^2 + 336T\tilde{A}^2 - 385\tilde{A}^3)$.

Assume first that $g = 0$. Then substituting $T = 3$ into the polynomials $g_4, \ldots, g_7$ and using *minAssChar* of Singular [15] we obtain Conditions (33) and (34) and a particular case of (α). Similarly, substituting $T = -2$ we obtain Conditions (24), (32) and, again, a particular case of (α). The case $4 + 2T - 5\tilde{A} = 0$, that is, the case when (34) holds, is considered above. Thus, there remains to consider the case when

$$h := 236 + 258T + 78T^2 + 4T^3 - 840\tilde{A} - 588T\tilde{A} - 84T^2\tilde{A} + 987\tilde{A}^2 + 336T\tilde{A}^2 - 385\tilde{A}^3 = 0.$$

Computing a Groebner basis of the ideal $\langle \bar{g}_5, \bar{g}_6, \bar{g}_7, h, 1-wd\rangle \subset \mathbb{C}[w, \tilde{A}, \tilde{L}, T]$ (here d is polynomial (33)) we see that it is equal to $\{1\}$. According to the Hilbert Nullstellensatz this means that the polynomial d vanishes on the variety of $\langle \bar{g}_5, \bar{g}_6, \bar{g}_7, h\rangle$, and the case when $d = 0$ we have considered above.

So, we now can assume that $g \neq 0$. We compute the resultants Θ_1 of Γ_1/g and Γ_2/g and Θ_2 of Γ_1/g and Γ_3/g with respect to $\tilde{A}$. Θ_1 and Θ_2 are polynomials in T. To find the zero set of Θ_1 and Θ_2 we tried to factorize these polynomials, but we were unable to perform this task with our computer facilities. However it is known that rings of univariant polynomials over a field are principal ideal domains and, therefore, for any polynomials $h_1, \ldots, h_s \in k[x]$ $\langle h_1, \ldots, h_s\rangle = \langle h\rangle$, where h is the greatest common divisor of $h_1, \ldots, h_s$ (see, e.g., [17]). Computing the greatest common divisor of Θ_1 and Θ_2 we obtain a polynomial $\Theta \in \mathbb{R}[T]$, such that $\Theta = \theta_1\theta_2\theta_3\theta_4\theta_5$, where $\theta_1 = (-5+T)^3(-4+T)^8(-3+T)^{89}(-2+T)^2(1+T)^2(2+T)^{89}(3+T)^8(4+T)^3(-5+2T)^4(-1+2T)^{99}(3+2T)^4(-7+3T)^2(-5+3T)^2(-4+3T)^5(-2+3T)^6(-1+3T)^6(1+3T)^5(2+3T)^2(4+3T)^2(-5+4T)^3(1+4T)^3(-7+5T)(-4+5T)(-1+5T)(2+5T)$, $\theta_2 = (-5927364 - 814032T + 3661635T^2 - 7522310T^3 + 8328915T^4 - 5481312T^5 + 1827104T^6)^2$ and θ_3, θ_4, θ_5 are polynomials in T of degrees 22, 30, and 100, respectively.

The roots of θ_1 are -4; -3; -2; -1; 2; 3; 4; 5; $5/2$; $1/2$; $-3/2$; $7/3$; $5/3$; $4/3$; $2/3$; $1/3$; $-1/3$; $-2/3$; $-4/3$; $-1/4$; $5/4$; $-2/5$; $7/5$; $4/5$; $1/5$.

We consider in detail the case $T = -4$. Substituting this value into the polynomials $g_4, \ldots, g_7$ and making use of *minAssChar* of Singular [15] we find that the minimal associate primes of $I = \langle g_4, \ldots, g_7\rangle$ are the ideals

$$I_1 = \Big\langle 2a_4\tilde{A} - a_4\tilde{L} - 22\tilde{A}\tilde{L} + 6\tilde{L}^2 - 5a_4, 2a_2\tilde{A} + 6\tilde{A}^2 - a_2\tilde{L} - 5\tilde{A}\tilde{L} + \tilde{L}^2 - 5a_2 - 27\tilde{A} + 11\tilde{L},$$

$$a_2\tilde{L}^2 + 3\tilde{A}\tilde{L}^2 - \tilde{L}^3 + 11a_2\tilde{L} + 33\tilde{A}\tilde{L} - 17\tilde{L}^2 - 6a_4\Big\rangle,$$

$$I_2 = \Big\langle \tilde{L} + 2, \tilde{A} + 1, 4a_2 - 3a_4 - 16\Big\rangle,$$

$$I_3 = \Big\langle \tilde{L}, \tilde{A}, 7a_2 + 2a_4\Big\rangle.$$

Equaling the polynomials defining I_1 to zero and resolving the obtained system with respect to a_2 and a_4 we obtain

$$a_2 = \frac{27\tilde{A} - 6\tilde{A}^2 - 11\tilde{L} + 5\tilde{A}\tilde{L} - \tilde{L}^2}{2\tilde{A} - \tilde{L} - 5}, \qquad a_4 = \frac{2\left(11\tilde{A}\tilde{L} - 3\tilde{L}^2\right)}{2\tilde{A} - \tilde{L} - 5}$$

(when $2\tilde{A} - \tilde{L} - 5 = 0$ the variety of I_1 is the empty set). Along with (28)–(30) the latter condition yields a particular case of (α).

Using the equations defining the ideal I_2 we have

$$\tilde{A} = -1, \qquad \tilde{L} = -2, \qquad a_4 = \frac{1}{3}(4a_2 - 16).$$

That yields Case (1) of (β). Under this condition system (14) has a solution of the form

$$y = -x + \cdots \tag{37}$$

defined implicitly by the equation

$$3x + 2x^2 + 3y + 2xy + 2y^2 = 0$$

and, therefore, the corresponding system of differential equations has a center. Finally, the defining equations of the ideal I_3 yield

$$\tilde{A} = 0, \qquad \tilde{L} = 0, \qquad a_4 = -\frac{7}{2}a_2,$$

that is, Case (2) of (β). In this case a solution to (14) of form (37) is given implicitly by

$$6x + 8x^2 + 3x^3 + 6y + 8xy + 3x^2y + 8y^2 + 3xy^2 + 3y^3 = 0.$$

Similarly, one can check that the remaining roots of θ_1 give Conditions (3)–(41) of (β) and using Theorem 2 verify that all these conditions also yield a center for corresponding systems (18).

In the remaining cases the roots of the polynomial Θ are irrational. Therefore, according to Proposition 1, for these values of T the system can have only a trivial center. However, due to Lemma 1 if T is irrational then the system has a trivial center only when Condition (α) holds. ∎

APPENDIX

Below is the MATHEMATICA code which we used to compute the first seven focus quantities of system (18) (the code is a slightly changed code from [12]). Here A and L stand for $\tilde{A}$ and $\tilde{L}$, respectively.

```
eq1 = (x + 1)*y; eq2 = r6 + r3 y + (T + 1)*y^2;
r6 = -x*(1 + a1*x + a2*x^2 + a3*x^3 + a4*x^4 + a5*x^5);
r3 = A0*x + A*x^2 + L*x^3;
z1, z2 = eq1,eq2;
u1 = (x^2 + y^2 + Sum[Sum[p[j,i - j] x^j y^(i - j),j,0,i],i,3,16]);
u2 = (D[u1,x]*z1 + D[u1, y]*z2 - Sum[g[i - 1](x^2 + y^2)^(i), i,2,8]);
u3 = CoefficientList[u2 /. x -> x*s, y -> y*s,s];
Do[w[i] = u3[[i + 1]], i, 3, 16];
Do[w[i] = CoefficientList[w[i], y] /. x -> 1,i,3,16];
Do[t[2*i + 1] = Solve[Table[w[2*i + 1][[j]] == 0, j,1,Length[w[2*i + 1]]],
Table[p[j, 2*i + 1 - j], j, 0, 2*i + 1]][[1]], i, 1, 7];
Do[t[2*i + 2] = Solve[Table[w[2*i + 2][[j]] == 0, j, 1, Length[w[2*i + 2]]],
Union[Table[p[j, 2*i + 2 - j], j, 1, 2*i + 2], g[i]]][[1]], i,1,7];
Do[Do[g[i] = Expand[g[i] /. t[2*i + 3 - j]], j,1,2*i],i,1,7];
Do[p[0, 2*j] = 0, j, 2, 8]; Do[g[i] = Factor[g[i]], i,1,7]
```

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